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BOUNDEDNESS OF PERIODIC SOLUTIONS OF ULTRA-HYPERBOLIC BOUNDARY VALUE PROBLEM WITH FOUR INDEPENDENT VARIABLES

U.E Obasi[1*], E.O Eze[2], and S.O Maliki[3]

[123]**Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.**

***Corresponding author:** sirurchobasi@gmail.com

ABSTRACT

In this paper, boundedness of periodic solutions of ultra-hyperbolic boundary value problem was investigated using energy method. Periodic solutions and test for periodicity were established using separation of variable method, analytical technique and Fourier series method. By establishing appropriate energy estimate and energy type functional that is non-negative and non-increasing function of time, boundedness of periodic solutions of ultra-hyperbolic boundary value problem was obtained. Furthermore, numerical simulations were used to demonstrate the behavior of the solutions extending known results in literature. Application of our results can be seen in wave propagation where wave transfer energy from one point to another.

Keywords: Boundedness, Energy method, Periodic Solution, Ultra-hyperbolic equation, Boundary value problem

1. Introduction

Consider the second order linear partial differential equation of the Ultra-hyperbolic type of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0 \ , \qquad (1.1)$$

With the following boundary and initial conditions

$u(0,y,z,t)=0$,$u(L_x,y,z,t)=0,\ \ 0\le x\le L_x \ \ t\ge 0$

$u(x,0,z,t)=0, u\big(x,L_y,z,t\big)=0, 0\le y\le L_y; u(x,y,0,t)=0, u(x,y,L_z,t)=0, 0\le z\le L_z$

$u(x,y,z,0)=f(x,y,z), \frac{\partial u}{\partial t}(x,y,z,0)=g(x,y,z)$ where L_x represent the length of the domain in the x direction, L_y represent the length of the domain in the y direction, L_Z represent the length of the domain in the z direction, u is the dependent variable, x,y,z,t are the independent variables, $\frac{\partial^2}{\partial x^2},\frac{\partial^2}{\partial y^2},\frac{\partial^2}{\partial z^2},\frac{\partial^2}{\partial t^2}$ represent second order partial derivatives with respect to x,y,z and t. u is periodic if $u\big(x+T_x,y+T_y,z+T_z,t+T\big)=u(x,y,z,t)$. $u\colon \Omega\subset\mathbb{R}^4\to\mathbb{R}$. Ω is the domain of u. For the given four independent

variables, the solution of (1.1) is said to be bounded if energy estimate is non-increasing and unbounded if any unbounded energy estimate exists. The wavelike form of equation (1.1) is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial t^2} \quad (1.2)$$

where $\frac{\partial^2 u}{\partial z^2}$ represent the frictional force. The frictional force is negligible so that there will be no loss of energy in the wave propagation. Equation (1.1) resembles the classical wave equation which has led to number of developments due to its characteristics (Frit, 1938). The applications of ultra-hyperbolic boundary value problem can be seen in the modelling of space time (Wang *et al*, 2022), designing of harbours and dams, earthquake analysis (electrodynamic wave propagation) and design of antennas (electromagnetic wave propagation) (Groenenboom, 1983). The problem can also be used in the study of symmetric spaces and elliptic differential operators (Zhang *et al*, 2022). Due to the characteristics of the ultra-hyperbolic boundary value problem, many authors have studied the problem producing sound results. For instance, see (Walter and Steven, 2008; Helgason, 1959; Owen, 1960; Golgelegen, 2023) and there references therein. Boundedness of partial differential equation is a qualitative property that ensure that solution do not grow indefinitely, remaining within a finite range. For partial differential equation of the ultra-hyperbolic type, boundedness is generally addressed by imposing specific boundary conditions on characteristic surfaces. Few authors have worked on boundedness of ultra-hyperbolic boundary value problem producing sound results. For instance, (Kostomarov, 2004) considered bounded of ultra-hyperbolic equation in a half space by formulating Cauchy problem for the equation. In (Klyuchnvk and Knut, 2018), bounded solutions of hyperbolic boundary value problem in a strip was investigated using zero-order coefficient. However, the use energy method to investigate boundedness of ultra-hyperbolic boundary value problem of four independent variables is rare in literature to the best of our knowledge. Other results on boundedness of partial differential equation can be seen in (Logan, 2004 and Akrid *et al*, 2024).

Motivated by the above literature, the objective of this paper is to investigate boundedness of periodic solutions of equation (1.1) using energy method. In section 2, we introduce some theorems and methods which will help us to obtain our main results. The analysis will be shown in section 3 while the paper is concluded in section 5 with a numerical simulation in section 4 to demonstrate the behavior of the system.

2. Theory and Methods

Definition 2.1 (Separation of variable method) Separation of variables is a method of solving ordinary and partial differentiation. For a partial differential equation in a function $\phi(x, y, \dots)$ and variables x, y, ..., separation of variables can be applied by making a substitution of the form

$$\phi(x, y, \dots) = X(x)Y(y) \dots, \quad (2.1)$$

breaking the resulting equation into a set of independent ordinary differential equations, solving these for $X(x), Y(y), \dots$, and then plugging them back into the original equation. Separation of variables was used by L'Hospital in 1970. It is useful in solving equations arising in Mathematical Physics such as Laplace's equation, the Helmholtz differential equation and the Schrodinger equation (Arfken, 1985).

Definition 2.2 (Periodic Solution) A solution $u(x, y, z, t)$ to a partial differential equation with four independent variables is said to be periodic in one or more variables if there exist $T > 0, T_x > 0, T_y > 0$ and $T_z > 0$ such that $u(x, y, z, t + T) = u(x, y, z, t), u(x + T_x, y, z, t) = u(x, y, z, t)$,

$u(x, y + T_y, z, t) = u(x, y, z, t)$, $u(x, y, z + T_z, t) = u(x, y, z, t)$ for $(x, y, z, t) \in \mathbb{R}$. T, T_x, T_y and T_z are the periods.

Definition 2.3 (Fourier series) This is a technique used to convert a complex wave function into a single sine or cosine function. Let $f(x)$ be a function in the interval $[-L, L]$, then Fourier series formula is given by $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right))$ where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$,
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \;\; n > 0$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \;\; n > 0$ are Fourier coefficients of f.

Theorem 2.4 (Superposition Principle) The principle of superposition state that if u_1 and u_2 are solutions of a linear PDEs then the linear combination $\alpha u_1 + \beta u_2$ is a solution where α and β are arbitrary constants (Chasnov, 2022).
Consider the ultra-hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad (2.2)$$

Suppose that $u = u_1(x, y, z, t)$ and $u = u_2(x, y, z, t)$ are solutions to equation (2.2). We consider a linear combination of u_1 and u_2 by letting $u = \alpha u_1 + \beta u_2$ where α and β are constants. The principle of superposition states that u is a solution of equation (2.2). To prove this, we compute

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial x^2} + \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial y^2} - \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial z^2} - \frac{\partial^2(\alpha u_1 + \beta u_2)}{\partial t^2} \qquad (2.3)$$

$$= \alpha\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 u_1}{\partial z^2} - \frac{\partial^2 u_1}{\partial t^2}\right) + \beta\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial z^2} - \frac{\partial^2 u_2}{\partial t^2}\right) = \alpha \times 0 + \beta \times 0 = 0$$

Since u_1 and u_2 were assumed to be solution of equation (2.2). We have therefore shown that any linear combination of solution to homogenous ultra-hyperbolic equation is also a solution.

Definition 2.5 Let V(x) be a function from $\mathbb{R}^n \to \mathbb{R}$, then $V(x)$ is a Lyapunov function if the following condition are satisfied.(i) $V(0) = 0$, (ii) $V(x) > 0$ for all $x \neq 0$, (iii) $V(x)$ is continuous and also have continuous derivative with respect to all components of x and (iv) $\dot{V}(x) \leq 0$

Definition 2.6 (Energy Method) Energy method is a powerful tool for analyzing and solving partial differential equations (PDEs) which are ubiquitous in various fields of Science and Engineering. The method is based on the concept of energy estimate which involves deriving bounds on the solution of a PDE using energy-like quantities. These energy estimates can be used to prove the existence, uniqueness and stability of solutions to PDEs (Lee, 2025).

Definition 2.7 Energy method for the wave equation in four variables (three spatial dimensions $x = (x_1, x_2, x_3)$ and one time dimension involves defining a total energy quantity for the system and showing that it is conserved or non-increasing quantity of time. For a homogenous linear wave equation $u_{tt} - c^2\Delta u = 0$, the total energy $\mathcal{E}(t)$ in the domain is given by

$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega}(u_t{}^2 + c^2|\nabla u|^2)dx$ (Evan, 2010). (2.4)

Definition 2.8 (Energy functional) Energy functional is a mathematical function that describes the total energy of a system as a function of its state variables. It is a powerful tool for analyzing and understanding the behavior of physical systems especially in fields like Physics.

Definition 2.9 Consider a real valued function $V: \mathbb{R}^n \to \mathbb{R}$ which is continuously differentiable. Then the continuous function $V(x)$ is said to be positive definite if $V(x) > 0$ and $V(0) = 0$ for all $x \neq 0$ and otherwise if it is negative definite.

Theorem 2.10 (Lyapunov Stability) Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function with $V(0) = 0$, $V(x) > 0$ in $D - \{0\}$ then (i) $x = 0$ is stable if $\dot{V}(x) \leq 0$, (ii) $x = 0$ is asymptotically stable if $\dot{V}(x) < 0$ and (iii) $x = 0$ is globally asymptotically stable if $V(x)$ is 'radially' bounded and $\dot{V}(x) < 0$.

D**efinition 2.11**The continuous function $V: R^n \to R$ is said to be (i) Positive Definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$. (ii) Negative Definite if $V(x) < 0$ for all $x \neq 0$ and $V(0) = 0$.
(iii)Positive Semi-definite if $V(x) \geq 0$ and it can vanish also for some$x \neq 0$. (iv) Negative Semi-definite if $V(x) \leq 0$ and it can also vanish for some $x \neq 0$.

Definition 2.12 A function is said to be bounded on the domain of definition $D(f)$ if there exist $M > 0$ such that $|f(x)| \leq M$ for all $x \in D(f)$. Also f is bounded if there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in D(f)$. This numbers m, M are called the lower and upper bound respectively.

3. Results and Discussion

3.1 Periodic Solutions of Ultra-hyperbolic equation

We consider the partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \ , \tag{3.1}$$

To obtain periodic solution of (3.1) we assume a solution of the form

$$u(x, y, z, t) = X(x)Y(x)Z(z)T(t) \tag{3.2}$$

Substituting equation (3.2) in equation (3.1) gives

$$YZT\frac{d^2X}{dx^2} + XZT\frac{d^2Y}{dy^2} - XYT\frac{d^2Z}{dz^2} - XYZ\frac{d^2T}{dt^2} = 0 \tag{3.3}$$

Dividing by $XYZT$ we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} \tag{3.4}$$

Setting equation (3.4) equal to k where k is the separation constant, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k \ \text{ and } \ \frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{T}\frac{d^2T}{dt^2} = k \tag{3.5}$$

In order to transform equation (3.5) into a system we use the following transformations
Let $X = \alpha x_0, Y = \beta y_0, \ Z = \gamma z_0$ and $T = \delta t_0$ where α, β, γ and δ are constant that maintain relationship between these variables.

Considering $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k$. Setting $Y = \beta y_0$, we have $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{\beta y_0}\frac{d^2(\beta y_0)}{dy^2} = k$ which gives $X''(x) - kX(x) = 0$. Also setting $X = \alpha x_0$ we have $\frac{1}{\alpha x_0}\frac{d^2(\alpha x_0)}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = k$ which gives $Y''(y) - kY(y) = 0$.

Considering $\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2}=k$. Setting $Z=\gamma z_0$ we have $\frac{1}{\gamma z_0}\frac{d^2(\gamma z_0)}{dz^2}+\frac{1}{T}\frac{d^2T}{dt^2}=k$ which gives $T''(t)-kT(t)=0$. Setting $T=\delta t_0$ we have that $\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{1}{\delta t_0}\frac{d^2(\delta t_0)}{dt^2}=k$ which gives $Z''(z)-kZ(z)=0$. Hence the system of equation is given by

$X''(x)-kX(x)=0$, $Y''(y)-kY(y)=0$, $Z''(z)-kZ(z)=0$ and $T''(t)-kT(t)=0$

Case 1: Considering for $k>0$, we have

$X(x)=c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x}$, $Y(y)=c_3e^{\sqrt{k}y}+c_4e^{-\sqrt{k}y}$, $Z(z)=c_5e^{\sqrt{k}z}+c_6e^{-\sqrt{k}z}$ and $T(t)=c_7e^{\sqrt{k}t}+c_8e^{-\sqrt{k}t}$. Where $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8$ are constants. Hence the general solution for case 1 is given by $u(x,y,z,t)=(c_1e^{\sqrt{k}x}+c_2e^{-\sqrt{k}x})(c_3e^{\sqrt{k}y}+c_4e^{-\sqrt{k}y})\left(c_5e^{\sqrt{k}z}+c_6e^{-\sqrt{k}z}\right)\left(c_7e^{\sqrt{k}t}+c_8e^{-\sqrt{k}t}\right)$. Applying the boundary conditions we have

$u(0.y,z,t)=0$ implies $c_1+c_2=0$. $u(L_x,y,z,t)=0$ implies $c_1e^{\sqrt{k}L_x}+c_2e^{-\sqrt{k}L_x}=0$. $u(x,0,z,t)=0$ implies $c_3+c_4=0$. $u\left(x,L_y,z,t\right)=0$ implies $c_3e^{\sqrt{k}L_y}+c_2e^{-\sqrt{k}L_y}=0$. $u(x,y,0,t)=0$ implies $c_5+c_6=0$. $u(x,y,L_z,t)=0$ implies $(c_5e^{\sqrt{k}L_z}+c_6e^{-\sqrt{k}L_z})=0$. Boundary condition for the independent variable x possess a non-trivial solution if and only if

$\begin{vmatrix}1 & 1\\ e^{\sqrt{k}L_x} & e^{-\sqrt{k}L_x}\end{vmatrix}=0$. $e^{-\sqrt{k}L_x}-e^{\sqrt{k}L_x}=0$ implies $\sqrt{k}=0$ since $L_x\neq 0$. The same result is true for y and z. This results are against the assumption for Case 1. Hence the solution is not acceptable.

Case II: for $k=0$, we have

$\frac{d^2X}{dx^2}=0,\frac{d^2Y}{dy^2}=0,\frac{d^2Z}{dz^2}=0$ and $\frac{d^2T}{dt^2}=0$ with solutions as $X=Ax+B, Y=Cy+D,$ $Z=Ez+F$ and $T=Gt+H$. Where B, D, F and H are constants. Therefore the required solution is

$$u(x,y,z,t)=(Ax+B)(Cy+D)(Ez+F)(Gt+H) \quad (3.6)$$

Using the boundary conditions we have $B=0, A=0, D=0, C=0, F=0$ and $E=0$. Hence only trivial solution is possible.

Case III: for $k<0: k=-\alpha^2, \alpha>0$. The differential equations are

$\frac{d^2X}{dx^2}+\alpha^2X=0,\frac{d^2Y}{dy^2}+\alpha^2Y=0,\frac{d^2Z}{dz^2}+\alpha^2Z=0$ and $\frac{d^2T}{dt^2}+\alpha^2T=0$ with solutions as $X(x)=c_1cos\alpha x+c_2sin\alpha x$, $Y(y)=c_3cos\alpha y+c_4sin\alpha y$, $Z(z)=c_5cos\alpha z+c_6sin\alpha z$ and $(x)=c_7cos\alpha t+c_8sin\alpha t$. . Hence the general solution is given by

$$u(x,y,z,t)=(c_1cos\alpha x+c_2sin\alpha x)(c_3cos\alpha y+c_4sin\alpha y)(c_5cos\alpha z+c_6sin\alpha z)(c_7cos\alpha t+c_8sin\alpha t) \quad (3.7)$$

Using the boundary conditions we have that for

$u(0,y,z,t)=0$, $c_1=0$; $u(L_x,y,z,t)=0$, $sin\alpha L_x=0$; $u(x,0,z,t)=0$, $c_3=0$;

$u\left(x,L_y,z,t\right)=0$, $sin\alpha L_y=0$; $u(x,y,0,t)=0$, $c_5=0$; $u(x,y,L_z,t)=0$, $sin\alpha L_z=0$

For $sin\alpha L_x=0$, $\alpha_n=\frac{n\pi}{L_x}$, $n=1,2,\ldots$, are the eigenvalues. The solution is

$$u_n(x,t)=sin\frac{n\pi x}{L_x}\left(A_ncos\frac{n\pi t}{L_x}+B_nsin\frac{n\pi t}{L_x}\right), \quad n=1,2,\ \ldots, \quad (3.8)$$

For $sin\alpha L_y=0$, $\alpha_m=\frac{m\pi}{L_y}$, $m=1,2,\ldots$, are the eigenvalues. The solution is

$$u_m(y,t)=sin\frac{m\pi y}{L_y}\left(C_mcos\frac{m\pi t}{L_y}+D_msin\frac{m\pi t}{L_y}\right), \quad m=1,2,\ \ldots, \quad (3.9)$$

For $sin\alpha L_z = 0,\ \alpha_g = \frac{g\pi}{L_z}, g = 1,2,\ldots$, are the eigenvalues. The solution is

$$u_g(z,t) = sin\frac{g\pi z}{L_z}\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right),\ g = 1,2,\ \ldots, \qquad (3.10)$$

Combining equations (3.8), (3.9) and (3.10) we have

$$u_p(x,y,z,t) = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)$$
$$\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \qquad (3.11)$$

Where $p = nmg$ and $A_n, B_n, C_m, D_m, E_g, F_g$ are constant coefficients

Using the superposition principle we have

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$
$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) \qquad (3.12)$$

Applying the initial conditions gives

$$u(x,y,z,0) = f(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} J_q \qquad (3.13)$$

where $J_q = (A_n)(C_m)(E_g)$. J_q is the half range Fourier sine series for A_n, C_m and E_g where $A_n = \frac{2}{L_x}\int_0^{L_x} f(x,y,z) sin\frac{n\pi x}{L_x} dx,\ C_m = \frac{2}{L_y}\int_0^{L_y} f(x,y,z) sin\frac{m\pi y}{L_y} dy$ and $E_g = \frac{2}{L_z}\int_0^{L_z} f(x,y,z) sin\frac{g\pi z}{L_z} dz$

Combining A_n, C_m and E_g we have

$$G_w = \frac{8}{L_x L_y L_z}\int_0^{L_x}\int_0^{L_y}\int_0^{L_z} f(x,y,z) sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y}\ sin\frac{g\pi z}{L_z} dxdydz \qquad (3.14)$$

$$\frac{\partial u}{\partial t}(x,y,z,t) = \left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)\left(\frac{-m\pi}{L_y} C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y} D_m cos\frac{m\pi t}{L_y}\right)$$
$$\left(\frac{-g\pi}{L_z} E_g sin\frac{g\pi \mathrm{t}}{L_z} + \frac{g\pi}{L_z} F_g cos\frac{g\pi t}{L_z}\right) + \left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(\frac{-g\pi}{L_z} E_g sin\frac{g\pi t}{L_z} + \frac{g\pi}{L_z} F_g cos\frac{g\pi t}{L_z}\right)$$
$$\left(\frac{-n\pi}{L_x} A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x} B_n cos\frac{n\pi t}{L_x}\right) + \left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right)\left(\frac{-n\pi}{L_x} A_n sin\frac{n\pi t}{L_x} + \frac{n\pi}{L_x} B_n cos\frac{n\pi t}{L_x}\right)$$
$$\left(\frac{-m\pi}{L_y} C_m sin\frac{m\pi t}{L_y} + \frac{m\pi}{L_y} D_m cos\frac{m\pi t}{L_y}\right) \qquad (3.15)$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = A_n D_m F_g \frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_m F_g B_n \frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_g B_n D_m \frac{n\pi}{L_x}\frac{m\pi}{L_y} \qquad (3.16)$$

$$\frac{\partial u}{\partial t}(x,y,z,0) = g(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$
$$\left(A_n D_m F_g \frac{m\pi}{L_y}\frac{g\pi}{L_z} + C_m F_g B_n \frac{g\pi}{L_z}\frac{n\pi}{L_x} + E_g B_n D_m \frac{n\pi}{L_x}\frac{m\pi}{L_y}\right) \qquad (3.17)$$

Hence the required solution is equation (3.12). Equation (3.12) is periodic in nature since the solution contain sinusoidal functions. The constant term and periodic term in (3.12) shows that the solution is periodic which is unstable in the sense that all solution starting sufficiently close to it depart from it with increasing time.

3.2 Test for Periodicity of Ultra-hyperbolic Equation using Fourier Series Method

We first establish the period for each variable. The formula for period of a periodic function given by $\frac{2\pi}{n}$ where n is the coefficient of x and 2π is the fundamental period.

$sin\frac{n\pi x}{L_x}$ is periodic in x with period $T_x = \frac{2\pi}{\frac{n\pi}{L_x}} = \frac{2L_x}{n}$, $cos\frac{n\pi t}{L_x}$ is periodic in t with period $T = \frac{2L_x}{n}$, $sin\frac{m\pi y}{L_y}$ is periodic in y with period $T_y = \frac{2L_y}{m}$ and $sin\frac{g\pi z}{L_z}$ is periodic in z with period $T_z = \frac{2L_z}{g}$. Next is to establish Fourier Series coefficient for each variable.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L_x}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L_x}\right) \quad (3.18)$$

For $f(x) = sin\frac{n\pi x}{L_x}$, Let $\frac{n\pi}{L_x} = c$, we have $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L} sincxdx = 0$, $a_n = \frac{1}{L}\int_{-L}^{L} sincxcoscxdx = 0$ and $b_n = \frac{1}{L}\int_{-L}^{L} sincxsincxdx = \frac{1}{L}\int_{-L}^{L} sin^2 cxdx = 1$

Therefore the Fourier series for variable x in given by

$$f(x) = \sum_{n=1}^{\infty} sin\frac{n\pi x}{L_x} \qquad n = 1,2,\ldots \quad (3.19)$$

Similarly for the function $f(y) = sin\frac{m\pi y}{L_y}$ and $(z) = sin\frac{g\pi z}{L_z}$, the Fourier series is given by

$f(y) = \sum_{m=1}^{\infty} sin\frac{m\pi y}{L_y}$ $m = 1,2,\ldots$ and $f(z) = \sum_{\alpha=1}^{\infty} sin\frac{g\pi z}{L_z}$ $g = 1,2,\ldots$

For the function $f(t) = cos\frac{n\pi t}{L_x}$, the Fourier series is gven by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L_x}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L_x}\right) \quad (3.20)$$

Let $\frac{n\pi}{L_x} = c$, then the coefficients are given by $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt = \frac{1}{L}\int_{-L}^{L} cosctdt = 0$, $a_n = \frac{1}{L}\int_{-L}^{L} cosctcosctdt = \frac{1}{L}\int_{-L}^{L} cos^2 ctdt = 1$ and $b_n = \frac{1}{L}\int_{-L}^{L} cosctsinctdt = 0$

Therefore the Fourier series is given by

$$f(t) = \sum_{n=1}^{\infty} cos\frac{n\pi t}{L_x} \quad \text{for } n = 1,2.3, \ \ldots \quad (3.21)$$

Since each variable has a Fourier series representation at a particular frequency and the Fourier coefficient are not all zeros, then the solution of ultra-hyperbolic equation exhibits periodicity at that frequency. To show that the solution of ultra-hyperbolic equation for four independent is periodic we established that

1. $u(x + T_x, y, z, t) = u(x, y, z, t)$. Considering $sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$ we have

$$sin\frac{n\pi(x+T_x)}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (3.22)$$

Note that $\sin(A + B) = sinAcosB + cosAsinB$. Let $\frac{n\pi x}{L_x} = A$ and $\frac{n\pi T_x}{L_x} = B$

$$sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) = sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x} \quad (3.23)$$

Equation (3.23) becomes

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = \left(sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}$$

$$= sin\frac{n\pi x}{L_x} cos\frac{n\pi T_x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} + cos\frac{n\pi x}{L_x} sin\frac{n\pi T_x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (3.24)$$

Since $T_x = \frac{2L_x}{n}$, $cos\frac{n\pi T_x}{L_x} = \cos 2\pi = 1$. Also $sin\frac{n\pi T_x}{L_x} = \sin 2\pi = 0$. Then we have

$$\sin\left(\frac{n\pi x}{L_x} + \frac{n\pi T_x}{L_x}\right) sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} = sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z} \quad (3.25)$$

$$u(x + T_x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{g=1}^{\infty} sin\frac{n\pi x}{L_x} sin\frac{m\pi y}{L_y} sin\frac{g\pi z}{L_z}\left(A_n cos\frac{n\pi t}{L_x} + B_n sin\frac{n\pi t}{L_x}\right)$$

$$\left(C_m cos\frac{m\pi t}{L_y} + D_m sin\frac{m\pi t}{L_y}\right)\left(E_g cos\frac{g\pi t}{L_z} + F_g sin\frac{g\pi t}{L_z}\right) = u(x,y,z,t) \quad (3.26)$$

The same results is true for y, z and t.

3.3 Boundedness of Solution to Ultra-hyperbolic equation using Energy method

Consider the ultra-hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2} \quad (3.27)$$

Multiplying equation (3.27) by u_t and integrating over the domain Ω gives

$$\int_\Omega u_t(u_{xx} + u_{yy} - u_{zz})d\Omega = \int_\Omega u_t u_{tt}\, d\Omega \quad (3.28)$$

Integrating the LHS of equation (3.32) we have

$\int_0^{L_x} u_t u_{xx} dx = -\frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_x}^2\, dx$, $\int_0^{L_x} u_t u_{yy} dy = -\frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_y}^2\, dy$ and

$\int_0^{L_x} u_t u_{zz} dz = -\frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_z}^2\, dz$. Substituting gives

$$\int_\Omega u_t u_{tt}\, d\Omega = -(\frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_x}^2\, dx + \frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_y}^2\, dy - \frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_z}^2\, dz) \quad (3.29)$$

But $\int_\Omega u_t u_{tt}\, d\Omega = \frac{1}{2}\frac{d}{dt}\int_\Omega (u_t)^2\, d\Omega$. $\frac{1}{2}\frac{d}{dt}\int_\Omega (u_t)^2\, d\Omega = -\varphi(x,y,z)$

where $\varphi = \left(\frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_x}^2\, dx + \frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_y}^2\, dy - \frac{1}{2}\frac{d}{dt}\int_0^{L_x} {u_z}^2\, dz\right)$. The energy functional is given by

$$E(t) = \frac{1}{2}\int_\Omega {u_t}^2\, d\Omega \quad (3.30)$$

From $\frac{1}{2}\frac{d}{dt}\int_\Omega (u_t)^2\, d\Omega = -\varphi(x,y,z)$, we have $\frac{dE}{dt} \le 0$ which by integration gives

$$E(t) \le K \quad (3.31)$$

where $K > 0$ is a constant. Equations (3.30) and (3.31) show that energy is non-negative and non-increasing function of time t. Due to the positivity of the energy and zero initial and boundary conditions, then the solution is zero. Equation (3.38) is the energy estimate for the ultra-hyperbolic boundary value problem. The estimate shows that energy is bounded. In other to estimate the bounds of solution, we have $\int_0^t \frac{dE(s)}{ds} ds \le 0$ implies $E(t) - E(0) \le 0$. This gives

$$E(t) \le E(0) \quad \text{for all } t \ge 0 \quad (3.32)$$

Equation (3.39) show that $0 \le E(t) \le E(0)$. This boundedness of energy show that the solution to ultra-hyperbolic is bounded.

4. Numerical Simulations of Ultra-hyperbolic equation

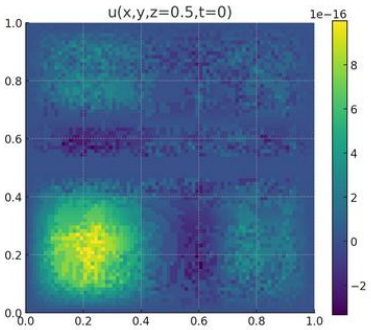


Figure 1**.** Diagram of periodic solution of ultra-hyperbolic equation for $t = 0$

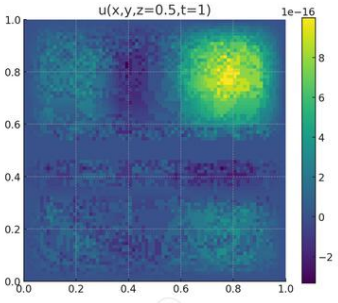


Figure 2**:** Diagram of periodic solution of ultra-hyperbolic equation for $t = 1$

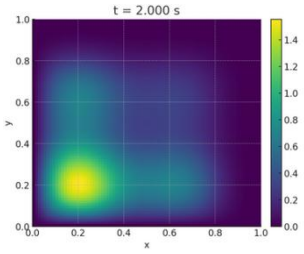


Figure 3**:** Diagram of periodic solution of ultra-hyperbolic equation for $t = 2$

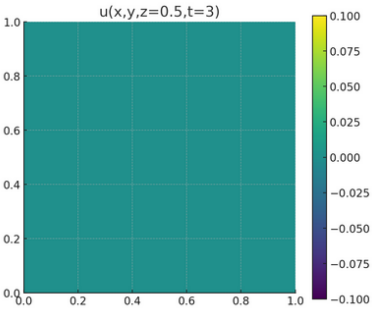


Figure 4: Dragram showing periodic solution of ultrahyperbolic equation for $t = 3$

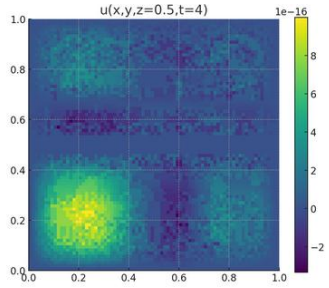


Figure 5**:** Diagram of periodic solution of ultra-hyperbolic equation for $t = 4$

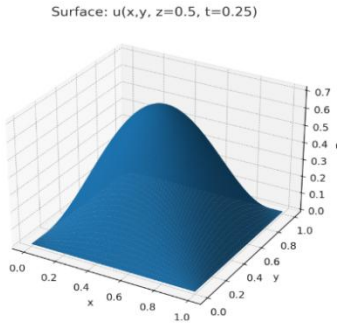


Figure 6. Diagram of periodic solution of ultra-hyperbolic equation for $t = 0.25$.

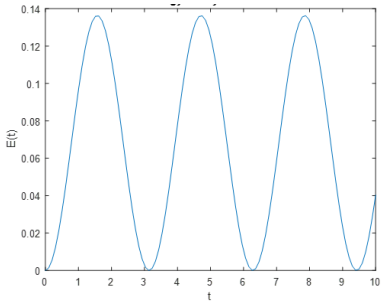


Figure 7**.** Diagram showing the energy of the system

4.1 Discussion of Numerical Behaviour of the Solutions

The behavior of the solutions are explained as follows:
Figure 1 describes the solution of ultra-hyperbolic equation for $t = 0$. In this case, the pattern of movement begins at its initial phase. This behavior describes the starting point of the wave propagation.
Figure 2 describes the solution of ultra-hyperbolic equation for $t = 1$. In this case, the pattern shift from initial position to another position maintaining the same amplitude and spatial dimension. This shift explain transfer of energy from one point to another. Figure 3 describes the solution of ultra-hyperbolic equation for $t = 2$. In this case, the pattern continues its oscillation with the same amplitude.
Figure 4 describes the solution of ultra-hyperbolic equation for $t = 3$. In this case, the solution disappear without no pattern of movement. This behavior explain the unstable nature of the system which may arise as a result of the mixed-signature nature of the model. Figure 5 describes solution of ultra-hyperbolic equation for $t = 4$. In the case, the pattern return exactly to its starting position. This behavior shows that the solution of ultra-hyperbolic equation is periodic.
Figure 6 describes the periodic nature of the ultra-hyperbolic equation which occurred as a result of changes in the values of the independent variables. Figure 7 describes energy function that oscillates periodically

with energy level repeating itself over time. The amplitude represent the maximum energy level of the system. The behaviour represent energy of a mechanical system such as simple pendulum. This behaviour can also represent sound wave or light wave that exhibit oscillatory behavior.

5. Conclusion

From our results, energy method was effective in establishing boundedness of ultra-hyperbolic boundary value problem. The Fourier series method was used to guarantee periodic solutions while the boundedness criteria play a central role in preventing blow-up phenomena. The advantages of energy method are that it provide a priori bounds on the solution and can be applied to a wide range of partial differential equations including nonlinear and non-autonomous systems. This implication of our results is that in finite time, the system will not explode to infinity thereby ensuring control in the system behavior.

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